Continuum Modelling of Traffic Flow

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1 Introduction

We wish to consider the problem of modelling flow of vehicles within a traffic network. In the past, stochastic traffic models have been developed [13, 8] in order to investigate a wide range of traffic configurations. However, if the population is sufficiently large, these stochastic processes tend towards a continuum approach. Consequently, this approach is most useful when considering long, densely-populated roads, but becomes invalid as the traffic on the road becomes sparse.

A number of different continuum models have been proposed in the literature, many of which share certain properties. Typically, these models are concerned with the flow rate of traffic \( q \), the traffic density \( \rho \) and the average flow velocity \( v \). Many models also involve the idea of an equilibrium flow rate, which prescribes a relationship between the traffic density and the flow rate, denoted \( q_e(\rho) \), or equivalently between the traffic density and the mean speed, denoted \( v_e(\rho) \). Michalopoulos [7] gives a general form for this relationship as

\[
v_e(\rho) = u_f \left[ 1 - \left( \frac{\rho}{\rho_j} \right)^\alpha \right]^\beta,
\]

where \( u_f \) represents the unimpeded traffic speed, \( \rho_j \) represents the density at which traffic can no longer flow, and \( \alpha \) and \( \beta \) are positive constants that depend upon the characteristics of the road section in question. For example, setting \( \alpha = \beta = 1 \) returns the relationship between average traffic speed and density posed by Greenshields [4]. Other relationships have been considered, with Greenberg [3] proposing the form

\[
q_e(\rho) = a \rho \log \left( \frac{\rho_j}{\rho} \right).
\]

This form seems to be problematic, as the flow does not approach zero as the density approaches zero, but as continuum models are no longer valid in this limit, useful results may still be obtained.
The most elementary continuum traffic flow model was the first order model developed concurrently by Lighthill & Whitham [6] and Richards [11], based around the assumption that the number of vehicles is conserved between any two points if there are no entrances (sources) or exits (sinks). This produces a continuum model known as the Lighthill-Whitham-Richards (LWR) model, given as

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0,$$

which will be briefly demonstrated in a subsequent section. This model has been used to analyse a number of traffic flow problems. Notably, both Lighthill & Whitham and Richards used the model to demonstrate the existence of shockwaves in traffic systems. This particular model does suffer from several limitations, as noted by Lighthill & Whitham. The model does not contain any inertial effects, which implies that the vehicles adjust their speeds instantaneously, nor does it contain any diffusive terms, which would model the ability of drivers to look ahead and adjust to changes in traffic conditions, such as shocks, before they arrive at the vehicle itself.

In order to address these limitations, Lighthill & Whitham propose a second-order model of the form

$$\frac{\partial \rho}{\partial t} + c \frac{\partial \rho}{\partial x} + T \frac{\partial^2 \rho}{\partial t^2} - D \frac{\partial^2 \rho}{\partial x^2} = 0,$$

where $T$ is the inertial time constant for speed variation, $c$ is the wavespeed (obtained from the relationship between $q$ and $\rho$, and $D$ is a diffusion coefficient representing how vehicles respond to nonlocal changes in traffic conditions. Second-order models were not explored again for some time, until Payne [10] and Whitham [14] concurrently developed a second-order continuum model governing traffic flow, given by Payne as

$$\frac{\partial \nu}{\partial t} + \nu \frac{\partial \nu}{\partial x} = -\frac{\nu - \nu_v(\rho)}{T} - \mu \frac{1}{T} \frac{\partial \rho}{\partial x},$$

where $\mu = -\nu_v(\rho)/2$. However, Daganzo [2] demonstrated that the Payne model, as well as several other second-order models available in the literature, produced flawed behaviour for some traffic conditions. Specifically, it was noted that traffic arriving at the end of a densely-packed queue would result in vehicles travelling backwards in space, which is physically unreasonable. This is due to the isotropic nature of the models, as the behaviour of vehicles is influenced by vehicles behind them due to diffusive effects.

Aw and Rascle [1] were able to produce an anisotropic second-order model that averted the flaws noted by Daganzo, obtained by considering the convective derivative of some pressure-type function of the density, given as $p(\rho)$. This model took the form

$$\frac{\partial}{\partial t} (\nu + p(\rho)) + \nu \frac{\partial}{\partial x} (\nu + p(\rho)) = 0,$$

where $p \sim \rho^\gamma$ with $\gamma > 0$ near $\rho = 0$, and $\rho p(\rho)$ is strictly convex. Other nonequilibrium higher-order models have been developed by Ross [12], Michalopoulos [7],
Kühne [5] and Zhang [15], all of which have been applied to a range of traffic flow problems. The model proposed by Zhang is notable in that it is also anisotropic, and therefore avoids the problem of backwards traffic flow.

While there are many models present in the literature, we will concentrate on a mathematical investigation of the model proposed by Lighthill & Whitham [6] given in (2) with particular emphasis on considering the behaviour of shock fronts. We will subsequently consider the extensions proposed by Lighthill & Whitham, with particular reference to the change in shock behaviour due to the addition of diffusive and inertial terms.

2 Development of Model

We begin by considering a small interval of road, between $x = x_1$ and $x = x_2$. We assume that the number of vehicles in this portion of road is conserved. As such, at any point in time, the change in the number of cars within the interval is given by the number of cars entering at $x_1$ and leaving at $x_2$, as illustrated in Figure 1. This may be expressed as

$$\frac{\partial}{\partial t} \left[ \int_{x_1}^{x_2} \rho(x,t) \, dx \right] = q(x_1,t) - q(x_2,t).$$

This is the integral form of the conservation equations. By considering this equation as the interval becomes small, we obtain a partial differential equation that governs the traffic flow system, given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0,$$

which is the formulation obtained by Lighthill & Whitham and Richards, or the LWR model. We will see that this formulation admits weak solutions which are discontinuous, but still satisfy the integral form of the conservation law, given in (3).

In order to formulate the problem completely, we require some relationship between flow rate and traffic density. Obviously, if there are no cars present on the road, there will be no flow. This condition is not strictly necessary, as the model is not valid at low traffic densities, however it is a useful condition to have present. Secondly, we assume that there is some density $\rho = \rho_j$ such that traffic may no longer flow. This is the jam density. We also require that the flow rate must be positive in $0 < \rho < \rho_j$, and that the flow is continuous on $0 \leq \rho \leq \rho_j$. We also assume that the flow is concave, or that $q''(\rho) < 0$.

As such, we conclude that for some value of the density $\rho = \rho_m$, the flow rate must be at a maximum. We assume that this is the unimpeded flow rate on the road described by the relationship between the density and the flow rate. An example relationship may be seen in Figure 2.
Figure 1: A segment of road between $x = x_1$ and $x = x_2$. The flow in at $x_1$ is given by $q(x_1, t)$, and the flow out at $x_2$ is given by $q(x_2, t)$. The vehicle density in the region is given by $\rho(x, t)$. As the number of vehicles is conserved, the change in the total number of vehicles within the region, given by the rate of change of the integral of vehicle density over the road interval, is equal to the outwards vehicle flow at $x_2$ subtracted from the inwards vehicle flow at $x_1$.

Figure 2: A possible relationship between flow rate ($q$) and density ($\rho$). We see that there is no flow for $\rho = 0$ and $\rho = \rho_j$, while the flow rate is maximal at $\rho = \rho_m$, where $0 < \rho_m < \rho_j$. 
As we assume that \( q = q(\rho) \), the LWR model may be expressed as

\[
\frac{\partial \rho}{\partial t} + q'(\rho) \frac{\partial \rho}{\partial x} = 0. \tag{4}
\]

Therefore the behaviour of the traffic transfer equation will be wavelike, with a wavespeed dependent on the relationship between the flow rate and the density.

This model, along with some prescribed equilibrium relationship \( q = q(\rho) \), is sufficient to allow exploration of traffic flow behaviour for a range of different traffic situations. We will initially use this model to investigate the nature of the shock fronts produced by the system, and subsequently apply these ideas in order to investigate the optimal configuration for traffic lights in order to ensure that the traffic does not bank up indefinitely.

### 3 Shock Solutions

We consider a situation where traffic in the region \( x < 0 \) initially has density \( \rho = \rho_a \), and the traffic in the region \( x \geq 0 \) has density \( \rho = \rho_b \).

Using the method of characteristics on (4) gives the system

\[
\frac{d\rho}{ds} = 0, \quad \frac{dt}{ds} = 1, \quad \frac{dx}{ds} = q'(\rho),
\]

where \( s \) is a characteristic variable. Eliminating \( s \) gives

\[
\rho = \rho(x - q'(\rho_0)t),
\]

where \( \rho_0 \) is the initial value of \( \rho \) obtained from the Cauchy data. We see that the characteristics will take the form of straight lines along which \( \rho \) remains constant with gradient \( q'(\rho_0) \) in \( x-t \) space.

We subsequently have two cases to consider in order to determine the behaviour of the shockwaves in the solution. Firstly, we consider the case where \( q'(\rho_a) < q'(\rho_b) \), and secondly, we consider the case where \( q'(\rho_a) > q'(\rho_b) \). Characteristic plots for both cases may be found in Figure 3.

#### 3.1 Case 1: \( q'(\rho_a) < q'(\rho_b) \)

In the first case, we find that the characteristics take the form demonstrated in Figure 3(a). There is, however, a region defined by \( q'(\rho_a)t < x < q'(\rho_b)t \) in which there are no characteristics apparent. In this region, however, we know that

\[
\frac{x}{t} = q'(\rho).
\]

As we have an expression for \( q'(\rho) \), we can solve this expression to determine \( \rho \) for any value of \( x \) and \( t \) within this region. This will tend to produce a number
3.2 Case 2: \( q'(\rho_a) > q'(\rho_b) \)

In the second case, we find that the characteristics take the form shown in Figure 3(b). We note that there is a region where the rays overlap. In order to eliminate multivaluedness in our solution, we must find a shock solution within this region, where the density changes instantaneously from \( \rho_a \) to \( \rho_b \). The location of this shock is not provided by the partial differential equation, but rather must be
obtained from the original conservation law governing the system.

Using methods described in Ockenden et al. [9] for determining the jump conditions from the original conservation equation, we find that the shock speed is given by

$$\frac{dx_s}{dt} = \frac{[q]}{[\rho]},$$

where $[\rho] = \rho_a - \rho_b$ and $[q] = q(\rho_a) - q(\rho_b)$. Therefore, the shock front will take the form of a straight line in the $x$-$t$ plane with gradient $[q]/[\rho]$. The precise location of the shock front depends on the relationship between $q$ and $\rho$, but a typical example may be seen in Figure 5.

We see that the LWR model for traffic flow predicts the existence of shock fronts for some traffic configurations when there is a discontinuity in the traffic density. We can use this result to consider the behaviour of traffic flow in more complicated situations, such as a traffic light at a junction.

### 4 Traffic Light Flow

In order consider the behaviour of traffic near a traffic light at $x = x_L$, we consider $t = 0$ to be the point at which the light turns green. Initially, we have traffic with density $\rho = \rho_j$ in front of the traffic light in the region $x_j \leq x < x_L$. In the region $x < x_j$, traffic is flowing with density $\rho_i < \rho_m$, as the flow is not unimpeded.

When the light turns green, vehicles are able to leave the light entirely unimpeded. As such, we find that the boundary condition on the right-hand side of the domain is given by $\rho = \rho_m$ on the boundary, which moves along with the first vehicles to pass through the lights at speed $q(\rho_m)/\rho_m$. However, the boundary will move faster than any other behaviour in the problem in the interval of concern to the
Figure 6: The initial density distribution for traffic flow at a set of traffic lights. Setting \( \rho = \rho_m \) beyond the traffic lights simplifies the problem slightly, as we no longer need to consider the moving boundary.

As \( \rho_i < \rho_m \) and \( q(\rho) \) is concave, we know that \( q'(\rho_i) > 0 \). We also know that \( q'(\rho_j) < 0 \), \( q'(\rho_m) = 0 \) and that the \( q(\rho) \) is continuous on \( 0 \leq \rho \leq \rho_j \).

The characteristic diagram is shown in Figure 7, and may be divided into three phases. In the first phase, the shockfront generated at \( x = x_j \) travels backwards through the traffic with speed \( \frac{dx_s}{dt} = \frac{-q(\rho_i)}{\rho_j - \rho_i} \), which is clearly negative. Additionally, we also see the fanning behaviour shown in section 3.1 in the region between the interval containing \( \rho = \rho_j \) and the interval containing \( \rho = \rho_m \). The vehicles that move in an unimpeded fashion when the light turns green will travel forwards at a speed of \( \frac{q(\rho_m)}{\rho_m} \). A typical cross-section of the density profile in this phase may be seen in Figure 8(a).

In the second phase, the region in which \( \rho = \rho_j \) has disappeared completely, and all of the traffic has started to move. The shock front slows down, and starts to move back towards the traffic light with speed \( \frac{dx_s}{dt} = \frac{q(\rho_m) - q(\rho_i)}{(\rho_m - \rho_i)} \), where \( \rho_m^+ \) represents the density on the right-hand side of the shock front. As \( q(\rho) \) is continuous, we find that the shock front path is also continuous. A typical cross-section of the density profile in this phase may be seen in Figure 8(b).

Finally, in the third phase, the shock front has reached the traffic light. On the right-hand side of the light, we see a shock front between the region in which \( \rho = \rho_i \) and the region in which \( \rho = \rho_m \). This shock front occurs in the region \( x > x_L \). At this point in time, the traffic is travelling at the incoming flow rate \( q(\rho_i) \) until it reaches the shock front, at which point the traffic speeds up to the maximum flow rate. The shock front will continue to travel with speed \( \frac{dx_s}{dt} = \frac{q(\rho_m) - q(\rho_i)}{\rho_m - \rho_i} \), which will be positive as \( \rho_i < \rho_m \).
A typical cross-section of the density profile in this phase may be seen in Figure 8(c).

In order to ensure that the traffic lights are able to clear all of the vehicles that were waiting at the light in the region \( x_j \leq x < x_L \), we require that the shock front that originated at \( x = x_j \) must pass the traffic lights at \( x = x_L \). This will occur if the flow through the traffic light is greater than or equal to the number of cars that arrive in a full cycle of the lights. The number of vehicles arriving at the lights over a single red-green cycle is given by \( q(\rho_i)(\tau_r + \tau_g) \) where \( \tau_r \) is the time interval over which the light is red, and \( \tau_g \) is the time interval over which the light is green. As such, if this number of vehicles is permitted to travel through the traffic light in the interval when traffic is permitted to flow, the length of the stopped region in front of the traffic light will not increase over a red-green cycle.

Prior to the shock front originating at \( x_j \) reaching the traffic lights (i.e., during the first two phases), the flow through the intersection is clearly given by \( q(\rho_m) \), and therefore the number of cars that are permitted to pass through the intersection in this period is given by \( q(\rho_m)\tau_r \), assuming the shock front does not reach the intersection in that time interval. As such, in order to ensure that traffic does not build up at the intersection, we require that

\[
q(\rho_m)\tau_r \geq q(\rho_i)(\tau_r + \tau_g),
\]

which tells us that the amount of time that traffic is permitted to flow through the intersection must be given by

\[
\tau_g \geq \left( \frac{q(\rho_m)}{q(\rho_i)} - 1 \right) \tau_r.
\]

Additionally, we see from Figure 7 that once the shock front reaches the traffic lights, the flow through the intersection decreases from \( \rho_m \) to \( \rho_i \). As such, if we wish to ensure that the junction is used to maximum efficiency, we should ensure that the light changes from green to red at the point in time when the shock front reaches \( x = x_L \). This ensures that traffic flowing through the junction is always flowing at \( q = q(\rho_m) \), and that all of the traffic that builds up to density \( \rho = \rho_j \) when the light is red is allowed to pass through when the traffic is permitted to flow.

5 Second-Order Model of Traffic Flow

Lighthill & Whitham [6] also proposed a second-order model that could potentially be used in order to investigate traffic flow, given by

\[
\frac{\partial \rho}{\partial t} + q'(\rho) \frac{\partial \rho}{\partial x} + T \frac{\partial^2 \rho}{\partial t^2} - D \frac{\partial^2 \rho}{\partial x^2} = 0,
\]
Figure 7: Characteristic diagram for traffic moving at a traffic light. The black rays represent $\rho = \rho_m$. The blue rays represent $\rho = \rho_j$. The red rays represent $\rho = \rho_i$. The thick black curve denotes the shock front, while the green rays represent an expansion fan between $\rho_j$ and $\rho_m$. We see that the diagram may be divided into three distinct phases. In the first phase, not all of the traffic that was banked up before the light changed has begun to move. In the second phase, all of the traffic has begun to move, but not all of the traffic that was banked up before the light changed has passed through the light. In the third phase, all of the traffic that was originally banked up with density $\rho_j$ before the light changed has passed through the traffic light.
Figure 8: A typical density distribution for traffic moving from traffic lights in (a) the first phase, (b) the second phase, and (c) the third phase. Regions with $\rho = \rho_i$, $\rho = \rho_j$ and $\rho = \rho_m$ are denoted by red, blue and black lines respectively. The thick black line represents the shock front, and the green line represents the behaviour demonstrated by the expansion fan.
where \( T \) is the inertial time constant for speed variation, and \( D \) is a diffusion coefficient representing how vehicles respond to nonlocal changes in traffic conditions. While this model has since been superseded by more rigourously developed models, it is still instructive to consider the effect of adding these extra terms on the shock front that propagates through the traffic flow.

Initially, we will determine the stability of the resultant system under small perturbations around some density \( \rho_0 \). We therefore set

\[
\rho = \rho_0 + \tilde{\rho},
\]

where \( \tilde{\rho} \) is some small perturbation. The resultant linearised equation becomes

\[
\frac{\partial \tilde{\rho}}{\partial t} + \frac{q'(\rho_0)}{\partial x} + T \frac{\partial^2 \tilde{\rho}}{\partial t^2} - D \frac{\partial^2 \tilde{\rho}}{\partial x^2} = 0.
\]

Following Whitham, we see that

\[
\frac{\partial}{\partial t} \approx -\frac{q'(\rho_0)}{\partial x},
\]

which gives

\[
\frac{\partial \tilde{\rho}}{\partial t} + q'(\rho_0) \frac{\partial \tilde{\rho}}{\partial x} = \left[ -T(q'(\rho_0))^2 - D \right] \frac{\partial^2 \tilde{\rho}}{\partial x^2} = 0.
\]

We see that this will only produce stable, diffusive behaviour if

\[
-T(q'(\rho_0))^2 - D > 0,
\]

or

\[
\frac{D}{T} > q'(\rho_0)^2.
\]

As such, we see that the system is only stable for wavespeeds in the interval

\[-\sqrt{\frac{D}{T}} < q'(\rho_0) < \sqrt{\frac{D}{T}}.\]

Increasing the response time of the drivers, or increasing \( T \) clearly reduces this interval, while increasing \( D \), or how far the drivers look ahead in order to respond to traffic conditions, increases this interval. Therefore, as the drivers become more aware, or the vehicles become more responsive, the wave speed interval in which the system is stable increases.

In order to determine the effect of considering a second-order system on the shock front, we consider the form of a travelling wave solution to the governing equation. Defining

\[
X = x - Ut, \quad \rho = \rho(X), \quad q = Q(\tilde{\rho}),
\]

we find that the governing equation becomes

\[
(D - U^2T) \frac{d^2 \tilde{\rho}}{dX^2} = (Q'(\tilde{\rho}) - U) \frac{d\tilde{\rho}}{dX}.
\]

Integrating this expression with respect to \( X \) gives

\[
(D - U^2T) \frac{d\tilde{\rho}}{dX} = Q(\tilde{\rho}) - U \tilde{\rho} + A,
\]
where $A$ is some constant of integration. Now, we consider the case where $q(\rho) = u_f \rho (1 - \rho / \rho_j)$, suggested by Greensheild [4], which is equivalent to the form given in (1) with $\alpha = \beta = 1$. If this form for the relationship between density and vehicle flow is applied to the previous equation and rearranged, we obtain

$$\frac{d\bar{\rho}}{dX} = \alpha \bar{\rho}^2 + \beta \bar{\rho} + B,$$

(7)

where

$$\alpha = -\frac{u_f}{\rho_j(D - U^2T)}, \quad \beta = \frac{u_f - U}{D - U^2T},$$

and $B$ is a new arbitrary constant. Now, we wish to investigate a configuration similar to that shown in Figure 9, where the density on the left of the wave is given by $\rho_a$, and the density on the right of the wave is given by $\rho_b$. As such, we would expect $d\bar{\rho}/dX$ to be zero for these values. As such, we would expect the right-hand side of (7) to be zero for $\bar{\rho} = \rho_a$ and $\bar{\rho} = \rho_b$. From this information, we may determine values of $B$ and $U$ that produce a solution to this problem, and subsequently determine the speed of the travelling wave.

This is algebraically difficult, however. Instead, we solve the differential equation given in (7) using direct integration to obtain

$$\bar{\rho} = -\frac{\beta}{2\alpha} - \frac{\sqrt{\beta^2 - 4\alpha B}}{2\alpha} \tanh \left\{ \frac{\sqrt{\beta^2 - 4\alpha B}}{2} (X + C) \right\},$$

(8)

where $C$ is some constant that determines the offset of the wave, which may be set to zero. We see that the travelling wave will have a smooth front, as it follows a hyperbolic tangent curve. The effect of including the higher-order derivative terms is to smooth out the wavefront, which is what we would expect from including diffusive terms in the problem.

As the curve must join the densities $\rho_a$ and $\rho_b$, we know that the offset of the hyperbolic tangent curve must be given by $(\rho_a + \rho_b)/2$, as seen in Figure 9. We also see in Figure 9 that the width of the curve must be given by $\rho_b - \rho_a$. Using the equation for the offset, we know from (8) that

$$-\frac{\beta}{2\alpha} = \frac{\rho_a + \rho_b}{2}.$$
Using the values for $\alpha$ and $\beta$ and rearranging the resultant equation, we find that

$$U = u_f \left( 1 - \frac{\rho_a + \rho_b}{\rho_j} \right).$$

As such, we have obtained speed of the wavefront that will propagate when we have two regions of differing vehicle density, which must travel from the more dense region to the less dense region over time.

As such, we see that the effect of including second-order derivative terms into the traffic flow model to represent the inertia of the car and the driver reaction is to smooth out the wavefront obtained from the shock solution. We also see that, if the traffic responds too slowly or the driver does not look far enough ahead, the system will become unstable. Finally, we were able to determine an expression for the speed at which the smoothed wavefront in traffic density will travel.

6 Conclusions

A number of stochastic and deterministic models have been proposed in order to investigate traffic flow. Many of these models involve the idea of an equilibrium flow rate, which predicts the relation between traffic density and the rate of unimpeded traffic flow. A number of forms for the equilibrium flow rate have been proposed in the literature. The Lighthill-Whitham-Richards (LWR) model is a basic linear ordinary differential equation model of traffic flow that incorporates the idea of equilibrium flow rate.

We described a method of deriving this model and subsequently applied the method of characteristics in order to investigate the existence and behaviour of shock solutions within the LWR model. We determined that it is possible to obtain both discontinuous and continuous shockwave solutions, depending on the densities on either side of the shock and the equilibrium flow rate model in use. We applied the analysis to a system containing a traffic light, in order to determine the optimal time the traffic light should remain green in order to maximise traffic flow and avoid a buildup of traffic.

Additionally, we considered an extension to the LWR model in order to incorporate second-order effects. This model incorporated the idea that the speed of each car does not change instantaneously and that drivers could respond nonlocally, as they could look ahead and determine the condition of the road in front of them. We considered a travelling wave solution produced in this situation, and demonstrated that the system would be unstable if the driver did not look far enough ahead in order to compensate for the inertia of the car, but would otherwise be stable.
References


